Lecture 22 – Agenda & Examples

Agenda

1. Review Question
2. Jacobian Transformations
   a. $f_{U,V}(u,v) = f_{x,y}\left(h_1^{-1}(u,v), h_2^{-1}(u,v)\right)|\det(J)|$
   b. Jacobian matrix $J = \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u} & \frac{\partial h_1^{-1}}{\partial v} \\ \frac{\partial h_2^{-1}}{\partial u} & \frac{\partial h_2^{-1}}{\partial v} \end{bmatrix}$
   c. Limitation: Can only use if both transformations are 1-1
   d. Advantage: Works for multivariate transformations where the RV's don't have to be independent & identically distributed (otherwise, the MGF technique would work well)
3. Order Stats Definitions
   a. Order Stats – ordering of a collection of random variables (or real-world data)
   b. I.I.D. – mutually independent and identically distributed (each RV has the same probability distribution as the other RV's)
4. Distributions of $X_{(1)}$ and $X_{(n)}$- proofs using the distribution function transformation technique
   a. $F_{X_{(n)}}(x) = [F(x)]^n \rightarrow f_{X_{(n)}}(x) = n[F(x)]^{n-1}f(x)$
   b. $F_{X_{(1)}}(x) = 1 - [1 - F(x)]^n \rightarrow f_{X_{(1)}}(x) = n[1 - F(x)]^{n-1}f(x)$
5. Examples

Review

1. (Textbook Exercises 6.41 & 6.43) Let $Y_1, Y_2, \dots, Y_n$ be i.i.d. (independent and identically distributed) normal random variables with common distribution $Y_i \sim N(\mu, \sigma^2)$.
   a. Let $a_1, a_2, \dots, a_n$ be known constants. Find the density function of the linear combination $U = \sum_{i=1}^{n} a_i Y_i$.
   b. Find the density function of $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ (think about the sampling distribution of sample means from your introductory course…look familiar?)
   c. If $\sigma^2 = 16$ and $n = 25$, what is the probability that the sample mean $\bar{Y}$ takes on a value that is within one unit of the population mean μ? (i.e., find $P(|\bar{Y} - \mu| \le 1)$).

Lecture

1. (Textbook Exercise 6.68) Let (Y_1, Y_2) have joint density function
$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 8y_1y_2, & 0 \le y_1 < y_2 \le 1 \\ 0, & o.w. \end{cases}$$
and $U = \frac{Y_1}{Y_2}, V = Y_2$.
   a. Derive the joint density function for (U, V).

b. Show that U and V are independent.

2. Suppose that X_1 and X_2 are independently exponentially distributed random variables, both with mean $\frac{1}{\lambda}$. Define $U = \frac{X_1}{X_1+X_2}$. Find the $f_U(u)$ and properly label this distribution (hint: use a Jacobian transformation here by first defining a handy auxiliary transformation V and finding the joint density of (U,V). Then, you can integrate out V and have the marginal).
3. (Textbook, Examples 6.16 & 6.17) Electronic components of a certain type have a length of life Y, with probability density given by
$$f(y) = \begin{cases} \left(\frac{1}{100}\right) e^{-\frac{y}{100}}, & y > 0 \\ 0, & o.w. \end{cases}$$
(length of life measured in hours). Suppose that two such components operate independently and in series in a certain system (hence, the system fails when either component fails).
   a. Find the density function for X, the length of life of the system. Label this distribution.
   b. Suppose that instead now, the system operated in parallel (hence, the system does not fail until both components fail). Find the density for X, the length of life of the system. Can we label this distribution? What do we notice?
4. (Textbook, Exercise 6.73) Let Y_1 and Y_2 be independently and uniformly distributed over the interval (0,1). Find the variance of Y_2 and the variance of $Y_{(2)}$. What do you notice? Why do you think this is so?