Lecture 23 – Final Review

1. Consider the following function:

$$f(x) = \begin{cases} ky & 0 \le y < 5 \\ \frac{2}{5} - ky & 5 \le y \le 10 \\ 0 & y < 0 \text{ } or \text{ } y > 10 \end{cases}$$

   a. Find k such that this function is a valid pdf on the given support
   b. Find the cdf
   c. Find the P(2 ≤ x ≤ 8)
2. Derive the cdf of the U(a,b) distribution.
3. Consider $X{\sim}Gamma(2v, 2)$,
   a. There is a special distribution used to define gamma RV's with beta = 2. Properly label this distribution.
   b. What is the third moment of this distribution?
4. Prove that the exponential distribution exhibits the memoryless property, i.e. that $P(X \ge t + t_0 | X \ge t_0) = P(X \ge t)$.
5. Derive the MGF of the uniform distribution. How do we restrict t in this case?
6. Use this MGF of the exponential distribution to then derive the variance of the uniform distribution.
7. If $X{\sim}N(2{,}2)$, what is the 50% percentile of the distribution?
8. If $X{\sim}N(2{,}2)$, what is the maximum value of this distribution, and at what value of x is this obtained?
9. Find the Harmonic mean of the beta distribution. What are Harmonic means used for?
10. Many times when using the exponential distribution, we are interesting in measuring a rate. What we have considered in class so far is the central exponential (or exponential distribution without any shift, thus having the support start at 0 and go off to infinity). Consider now the shifted exponential distribution $f(x) = \lambda e^{-\lambda(x-\theta)}$ with new support $x > \theta$. Now, instead of starting the support at 0 we start at theta and go off to infinity. Derive the second moment of this distribution to be $\frac{2}{\lambda^2} + \frac{2\theta}{\lambda} + \theta^2$ using the techniques we've learned (hint: first use the substitution of $y = x - \theta$, then write the integral as the sum of 3 integrals, then use u-substitution on all 3 integrals to get gamma integrals and simplify).
11. Consider the following function:

$$f(x_1, x_2) = k(x_1 - x_2), \quad x_1 = 2{,}3{,}4 \;\; x_2 = 1{,}2$$

   a. Find k such that this is a valid pmf on the given support (a table could help)
   b. Find the marginal distributions of x_1 and x_2.
   c. Is $x_1 \perp x_2$?
   d. Find the $E[(X_1 + 1)^2 - 1 | X_2 = 1]$
12. Consider the following joint pdf:

$$f(x_1, x_2) = \frac{1}{\pi\sqrt{56}} \exp\left(-\frac{1}{7}\left(x_1^2 - \frac{x_1 x_2}{\sqrt{2}} + x_2^2\right)\right) \quad x_1, x_2 \in \mathbb{R}$$

   a. What is the $E[X_1^2]$?
   b. What is the $E[X_1^2 - 3X_1 | X_2 = 1]$?

c. Are X_1 and X_2 independent?
d. What is the $P(|X_2| < 2)$?
e. Let $U_1 = X_1 + X_2$ and $U_2 = X_1 - X_2$ Prove that the $Cov(U_1, U_2) = 0$ and then show that the $V[U_1 + U_2] = 16 + 4\sqrt{2}$ (hint: use the correlation that you know to replace (properly!) the covariance that you don't).
f. T/F, if $Cov[X, Y] = 0$, then $X \perp Y$?

13. Consider the typical Bayesian setup. We have our data, represented by the random variable $X|\lambda$ following a $Exp(\lambda)$ distribution. Now, we also have a prior, or prior information about our parameter. Here, we know that lambda varies from day to day according to a $Beta(\alpha, \beta)$ distribution.
    a. Show that $E[X] = \frac{\alpha+\beta-1}{\alpha-1}$
    b. Show that $V[X] = \frac{2(\alpha+\beta-1)(\alpha+\beta-2)(\alpha-1)-(\alpha-2)(\alpha+\beta-1)^2}{(\alpha-1)^2(\alpha-2)}$
14. (Text, 6.33) The proportion of impurities in certain ore samples is a random variable Y with a pdf given by $f(y) = \frac{3}{2}y^2 + y$ for $y \epsilon [0,1]$, 0 otherwise. The dollar value of such samples is $U = 5 - \frac{Y}{2}$. Find the pdf for U using the method of transformations. Don't forget the support!
15. Let X_1 and X_2 be independent random variables $X_i \sim Gamma(\alpha_i, \beta)$, $\alpha_i > 0, \beta > 0, i = 1,2$. Define $Y_1 = X_1 + 2X_2$ and $Y_2 = X_1$
    a. Are the transformations 1-1? Back-solve for X_1 and X_2 in terms of Y_1 and Y_2.
    b. Since this is a two-variable case, we need the Jacobian, J. What is J, det(J), and thus |det(J)|?
    c. Now, what is $f_{X_1,X_2}[h^{-1}(y_1, y_2)]$, i.e. the joint distribution of X_1 and X_2 evaluated at X_1 and X_2 as functions of Y_1 and Y_2? (Hint: you'll first need the joint distribution of $f(x_1, x_2)$)
    d. What is $f_{Y_1,Y_2}(y_1, y_2)$? Don't forget the support!
16. Consider $Y_1, Y_2, \ldots, Y_n$ independently and identically distributed (i.i.d.) $U(0, \frac{2\theta}{3})$ random variables.
    a. Find the $E\left[Y_{(n)}\right]$.
    b. Find $F\left[Y_{(1)}\right]$.
    c. Find the joint distribution of $Y_{(1)}$ and $Y_{(n)}$.
    d. Give a real-world example of when you would be concerned with the joint distribution of the min and max?
17. Derive the expectation of the Normal Distribution.