Lecture 17 – Agenda & Examples

Agenda

1. Review Questions
2. Marginal Distribution
   a. $p_1(x_1) = \sum_{x_2 \in S_2} p(x_1, x_2)$
   b. $f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2$
3. Conditional Distribution
   a. $p_{1|2}(x_1) = \frac{p(x_1, x_2)}{p_2(x_2)}$
   b. $f_{1|2}(x_1) = \frac{f(x_1, x_2)}{f_2(x_2)}$
4. Proof that any conditional distribution is a valid distribution (assuming the joint and marginal distributions are valid distributions).
5. These are distributions, so they should integrate/sum to 1 and we can find expectation!
6. A very special joint distribution is the Bivariate Normal Distribution
   $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$
   a. *Let X_1, X_2 be continuous random variables with the joint pdf*

   $$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right.\right.$$
   $$\left.\left. - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right)$$
   $$-\infty < \mu_1, \mu_2 < \infty, \qquad 0 < \sigma_1, \sigma_2, \qquad -1 < \rho < 1$$

   b. The marginal distribution of X_i is given by $N(\mu_i, \sigma_i^2)$ for i = 1,2
   c. The conditional distribution of $X_1 | X_2 = x_2$ is $N\left(\mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2 - \mu_2), \sigma_1^2(1 - \rho^2)\right)$ for all fixed $x_2 \in \mathbb{R}$
   d. The conditional distribution of $X_2 | X_1 = x_1$ is $N\left(\mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x_1 - \mu_1), \sigma_2^2(1 - \rho^2)\right)$ for all fixed $x_1 \in \mathbb{R}$
7. Examples

Review

1. Let X_1 and X_2 be continuous random variables with the following joint pdf:
   $$f(x_1, x_2) = \begin{cases} e^{-x_1}, & 0 < x_1 \text{ and } 0 < x_2 < 1 \\ 0, & o.w. \end{cases}$$
   a. Confirm this is a valid pdf over the given support
   b. Find the $P(.5 < X_1 + X_2 < 1)$.
2. Consider the following joint pmf of 2 random variables X, Y:
   $$p(x, y) = \frac{x + y + 1^x}{9} \text{ for } x \in \{0,1,2\} \, y \in \{0, c\}$$

      a. Find the value c such that this is a joint pmf on the given support.

      b. What is $P(X = 2)$?

      c. What is $P(X < 2)$?

      d. What is $P(X - Y^2 < 0)$?

      e. What is the $P(X + Y^2 - 3 < 2)$?

3. Suppose that Y_1 and Y_2 are uniformly distributed over the triangle shaded such as in Exercise 5.11 from your text.

      a. What is the joint pdf of Y_1 and Y_2, i.e. $f(y_1, y_2)$?

      b. Confirm that this is a valid pdf over the defined support

      c. What is the $P(Y_1 < 0)$?

      d. What is the $P\left(Y_1 < \frac{1}{2}\right)$?

      e. What is the $P\left(Y_1 \le \frac{3}{4}, Y_2 \le \frac{3}{4}\right)$?

Lecture

1. Consider the distribution $f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1 \\ 0, & o.w. \end{cases}$

      a. Find the marginal density function for Y_2

      b. For what values of y_2 is the conditional density $f(y_1|y_2)$ defined?

      c. What is the $P\left(y_2 > \frac{1}{2}\middle|y_1 = \frac{3}{4}\right)$?

2. Suppose that we have $S_1 = \{0,1,2\}$, $S_2 = \{-1,0,1\}$ and the joint distribution $p(x_1, x_2)$ defined as $p(x_1, x_2) = 0$ when $(x_1, x_2) = (0,-1), (0,1), (1,0), (2,-1), (2,1)$ and $p(x_1, x_2) = .25$ when $(x_1, x_2) = (0,0), (1,-1), (1,1), (2,0)$.

      a. Obtain the marginal pmf of X_1, $f_1(x_1)$.

      b. Prove that $f_1(x_1)$ is a valid distribution and find the $E[X_1]$.

      c. Find the conditional pmf $f_{1|2}(x_1)$ when $x_2 = -1$.

3. Consider the joint distribution of X_1, X_2, X_3, X_4
$f(x_1, x_2, x_3, x_4) = \lambda^4 e^{-\lambda(x_1+x_2+x_3+x_4)}$ for $0 < x_1, x_2, x_3, x_4 < \infty$, $\lambda > 0$

      a. Prove that the distribution is a valid distribution

      b. Find $f_3(x_3)$ and label this distribution.

      c. Find $f_{1,2,3}(x_1, x_2, x_3)$

      d. Find $f_{1,2,3|4}(x_1, x_2, x_3)$. Notice anything?

4. Assume that $(X_1, X_2) \sim N_2\left(2, 5, 4, 9, \frac{1}{2}\right)$

      a. What is $P(x_1 > 2)$?

      b. What is the $E[X_2^2]$?

5. Recall that two Bivariate Normal random variables X_1 and X_2 have the following joint pdf

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right)$$

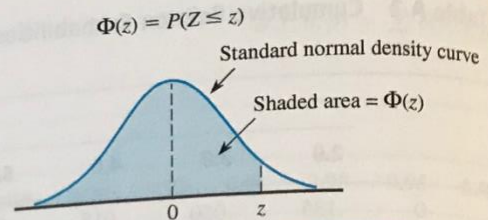What happens when the two random variables are independent?

6. Suppose that 2 random variables have the following joint pdf

$$f(x_1, x_2) = \frac{1}{8\pi\sqrt{3}} \exp\left(-\left(\frac{2}{3}\right)\left(\left(\frac{x_1 - 1}{2}\right)^2 - \left(\frac{x_1 - 1}{2}\right)\left(\frac{x_2 - 1}{4}\right) + \left(\frac{x_2 - 1}{4}\right)^2\right)\right)$$

Find the $E[X_1^2 | X_2 = x_2]$.

Table A.3 Standard Normal Curve Areas

$\Phi(z) = P(Z \le z)$

Standard normal density curve

Shaded area = $\Phi(z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0038
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3482
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

(continued)

Table A.3   Standard Normal Curve Areas (cont.)

$$\Phi(z) = P(Z \le z)$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998