Lecture 18 – Agenda & Examples

Agenda

1. Review Questions
2. A very special joint distribution is the Bivariate Normal Distribution $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$
   a. $Let\ X_1, X_2$ be continuous random variables with the joint pdf
   $$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right)$$
   $$-\infty < \mu_1, \mu_2 < \infty, \qquad 0 < \sigma_1, \sigma_2\,, \qquad -1 < \rho < 1$$
   b. The marginal distribution of X_i is given by $N(\mu_i, \sigma_i^2)$ for i = 1,2
   c. The conditional distribution of $X_1|X_2 = x_2$ is $N\left(\mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2 - \mu_2), \sigma_1^2(1 - \rho^2)\right)$ for all fixed $x_2 \in \mathbb{R}$
   d. The conditional distribution of $X_2|X_1 = x_1$ is $N\left(\mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x_1 - \mu_1), \sigma_2^2(1 - \rho^2)\right)$ for all fixed $x_1 \in \mathbb{R}$
3. Expectation Properties
   a. $E[g(X_1, \dots, X_n)] = \sum_{x_1 \epsilon S_1} \dots \sum_{x_n \epsilon S_n} g(x_1, \dots, x_n) p(x_1, \dots, x_n)$
   b. $E[g(X_1, \dots, X_n)] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_1 \dots dx_n$
   c. $E[c] = c$
   d. $E[cg(X_1, \dots, X_n)] = cE[g(X_1, \dots, X_n)]$
   e. $E\left[\sum_{i=1}^{k} g_i(X_1, \dots, X_n)\right] = \sum_{i=1}^{k} E[g_i(X_1, \dots, X_n)]$
4. Independence
   a. Continuous Case - X_1 and X_2 are independent iff $f(x_1, x_2) = f_1(x_1)f_2(x_2)\ \ \forall x_1, x_2$ (naturally this extends to the cdfs as well)
   b. Discrete Case - X_1 and X_2 are independent iff $p(x_1, x_2) = p_1(x_1)p_2(x_2)\ \ \forall x_1, x_2$ (naturally this extends to the cdfs as well)
   c. More generally, if for constants a,b,c,d (a < b, c < d) the pdf is positive only when $a \le x_1 \le b$ and $c \le x_2 \le d$, and 0 otherwise, then X_1 and X_2 are independent iff $f(x_1, x_2) = g(x_1)h(x_2)$ for any functions g and h.
5. If X_1 and X_2 are random variables, let g and h be real valued functions defined over the supports of X_1 and X_2, respectively. Then, if X_1 and X_2 are independent, then $E[g(X_1)h(X_2)] = E[g(X_1)]E[g(X_2)]$
6. If X_1 and X_2 are independent, then $Var[X_1 + X_2] = Var[X_1 - X_2] = Var[X_1] + V[X_2]$
7. Examples

Review

1. Suppose that we have $S_1 = \{0,1,2\}$, $S_2 = \{-1,0,1\}$ and the joint distribution $p(x_1, x_2)$ defined as $p(x_1, x_2) = 0$ when $(x_1, x_2) = (0,-1), (0,1), (1,0), (2,-1), (2,1)$ and $p(x_1, x_2) = .25$ when $(x_1, x_2) = (0,0), (1,-1), (1,1), (2,0)$.
   a. Obtain the marginal pmf of X_1, $f_1(x_1)$.
   b. Prove that $f_1(x_1)$ is a valid distribution and find the $E[X_1]$.
   c. Find the conditional pmf $f_{1|2}(x_1)$ when $x_2 = -1$.
2. Consider the joint distribution of X_1, X_2, X_3, X_4
   $f(x_1, x_2, x_3, x_4) = \lambda^4 e^{-\lambda(x_1+x_2+x_3+x_4)}$ for $0 < x_1, x_2, x_3, x_4 < \infty, \lambda > 0$
   a. Prove that the distribution is a valid distribution
   b. Find $f_3(x_3)$ and label this distribution.
   c. Find $f_{1,2,3}(x_1, x_2, x_3)$
   d. Find $f_{1,2,3|4}(x_1, x_2, x_3)$. Notice anything?
3. Assume that $(X_1, X_2) \sim N_2\left(2, 5, 4, 9, \frac{1}{2}\right)$
   a. What is $P(x_1 > 2)$?
   b. What is the $E[X_2^2]$?

Lecture

1. Recall that two Bivariate Normal random variables X_1 and X_2 have the following joint pdf

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right)$$

   What happens when the two random variables are independent?
2. Suppose that 2 random variables have the following joint pdf

$$f(x_1, x_2) = \frac{1}{8\pi\sqrt{3}} \exp\left(-\left(\frac{2}{3}\right)\left(\left(\frac{x_1-1}{2}\right)^2 - \left(\frac{x_1-1}{2}\right)\left(\frac{x_2-1}{4}\right) + \left(\frac{x_2-1}{4}\right)^2\right)\right)$$

   Find the $E[X_1^2 | X_2 = x_2]$.
3. Consider the joint density below:

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & x_1 > 0, x_2 > 0 \\ 0, & o.w. \end{cases}$$

   a. Are X_1 and X_2 independent?
   b. What is the $E[X_1^2 X_2^3]$?
   c. What is the $E[(X_1 - 4)^2 (X_2 + 1)]$?
   d. What is the $V[X_1 + X_2]$ using the variance shortcut formula? How about using the properties of independence? What do you notice?
   e. What is the $P(X_1 - X_2 < -3)$?

4. Suppose that X_1 and X_2 are independent χ^2 random variables with degrees of freedom v_1 and v_2, respectively.
    a. Find $E[X_1 + X_2]$
    b. Find $V[X_1 + X_2]$