Lecture 21 – Agenda & Examples

Agenda

1. Review Questions
2. Method of MGF
    a. Steps
        i. Write the MGF of U as a function of the MGF of the known distribution $M_U(t) = E[e^{tU}] = E\left[e^{h(X)t}\right]$
        ii. Identify the form of $M_U(t)$ as that of a known MGF
        iii. Connect this to identify the pdf of U
    b. Limitation: You may not always get a known version of the MGF for $M_U(t)$
    c. Advantage: Works best with independent & identically distributed RV's.
3. Note: For each pset question, please use the method detailed in the section that the question comes from.
4. Jacobian Transformations
5. Examples

Review

1. Consider the following joint density function
$$f(x,y) = \begin{cases} 2(1-x), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & o.w. \end{cases}$$
Find the pdf for $U = XY$. (hint: use the transformation method)
2. (Textbook Exercise 6.2) Let Y be a random variable with a density function given by
$$f(y) = \begin{cases} \left(\frac{3}{2}\right)y^2, & -1 \le y \le 1 \\ 0, & o.w. \end{cases}$$
    a. Find the pdf of $U_1 = 3Y$ (hint: use the density function approach)
    b. Find the pdf of $U_2 = 3 - Y$ (hint: use the density function approach)
3. (Textbook Exercise 6.27) Let Y have an exponential distribution with mean $\frac{1}{\lambda}$. Prove that $W = \sqrt{Y}$ has a Weibull pdf, i.e. $f(w) = 2\lambda w e^{-w^2\lambda}, w \ge 0$ (hint: use the transformation method)
4. (Textbook Exercise 6.28) Let Y have a Uniform(0,1) distribution. Show that $U = -2\ln(Y)$ has an exponential distribution with mean 2. (hint: use the transformation method)
5. (Textbook Exercise 6.29) The speed of a molecule in a uniform gas at equilibrium is a random variable X whose probability density function is given by
$$f(x) = ax^2e^{-bx^2}, x > 0$$
where $b = \frac{m}{2kT}$ k, T, m constants. Derive the distribution of $W = \frac{mX^2}{2}$, the kinetic energy of the molecule (hint: use the transformation method)

Lecture

1. (From Midterm Exam) Suppose that Y is a random variable with the MGF $m_Y(t)$, and W is given by $W = aY + b$
   a. Show that the MGF of W is $e^{tb}m_Y(at)$
   b. What is the $m_W(0)$?
   c. Use the MGF of W to prove that $E[W] = aE[Y] + b$ (hint: $m_Y'(ta) = am_Y'(ta)$).
2. (Textbook Exercises 6.41 & 6.43) Let $Y_1, Y_2, \dots, Y_n$ be i.i.d. (independent and identically distributed) normal random variables with common distribution $Y_i \sim N(\mu, \sigma^2)$.
   a. Let $a_1, a_2, \dots, a_n$ be known constants. Find the density function of the linear combination $U = \sum_{i=1}^{n} a_i Y_i$.
   b. Find the density function of $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ (think about the sampling distribution of sample means from your introductory course…look familiar?)
   c. If $\sigma^2 = 16$ and $n = 25$, what is the probability that the sample mean $\bar{Y}$ takes on a value that is within one unit of the population mean μ? (i.e., find $P(|\bar{Y} - \mu| \leq 1)$).