Lecture 13 – Agenda & Examples

Agenda

1. Review Questions
2. Gamma Function & Its Properties
   a. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ (for $\alpha > 0$)
   b. For any $\alpha > 1, \Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
   c. For any positive integer n, $\Gamma(n) = (n-1)!$
   d. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
   e. Evaluate the following:
      i. $\int_0^\infty x^3e^{-x}dx$
      ii. $\int_0^\infty x^4e^{-\frac{1}{2}x}dx$
      iii. $\int_0^\infty x^2e^{-\frac{1}{2}x^{\frac{3}{2}}}dx$
      iv. $\int_0^\infty xe^{-\frac{1}{3}x^2}dx$
3. Gamma Distribution $\left(X \sim Gamma(\alpha, \beta)\right)$
   a. Proof of valid pdf $\left(f(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}} \quad x \geq 0\right)$ with $\alpha, \beta > 0$
   b. Derivation of expectation and Variance ($E(X) = \alpha\beta$ and $V(X) = \alpha\beta^2$)
   c. MGF
4. Exponential Distribution $\left(X \sim Exp(\lambda)\right)$
   a. pdf $\left(f(x) = \lambda e^{-\lambda x} \quad x \geq 0\right)$ with $\lambda > 0$ (If $X \sim Gamma(1, \beta)$, look familiar?)
   b. cdf $\left(F(x) = 1 - e^{-\lambda x} \quad x \geq 0\right)$
   c. Expectation and Variance ($E(X) = \frac{1}{\lambda}$ and $V(X) = \frac{1}{\lambda^2}$)
   d. Memoryless Property $P(X \geq t + t_0 | X \geq t_0) = P(X \geq t) = e^{-\lambda t}$
5. Chi-Squared Distribution $\left(X \sim \chi^2(\nu)\right)$
   a. pdf $\left(f(x) = \frac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}x^{\left(\frac{\nu}{2}\right)-1}e^{-\frac{x}{2}} \quad x \geq 0\right)$ with $\nu = \text{d.f.}$ (If $X \sim Gamma(\frac{\nu}{2}, 2)$, look familiar?)
   b. Expectation and Variance ($E(X) = \nu$ and $V(X) = 2\nu$)
6. Examples

Review

1. Consider the following pdf
$$f(x) = \begin{cases} cx^2 + x & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$
   a. Find c such that the above function is a valid pdf on the defined support.
   b. Find $F(x)$
2. Consider the following pdf

$$f(x) = \begin{cases} .2 & -1 < x \leq 0 \\ .2 + cx & 0 < x \leq 1 \\ 0 & otherwise \end{cases}$$

a. Find c such that the above function is a valid pdf on the defined support.
b. Find $F(x)$
c. Find $P(X > .5 | X > .1)$

3. Derive the generalized form of the 4th moment of any uniform distribution
4. Suppose that $X \sim U(2,10)$, what is $E(X^4 - (X-2)^2)$

Lecture

1. Find the seventh moment if $X \sim Gamma(\alpha, \beta)$.
2. Find the 83rd percentile of a chi-square distribution with 2 degrees of freedom.
3. Find the k such that the following is a valid pdf:

$$f(x) = \begin{cases} \frac{k}{\sqrt{\pi}\sqrt{x}} e^{-k^2 x} & 0 \leq x \\ 0 & otherwise \end{cases}$$

4. Derive the expectation of the Chi-Square distribution.
5. Suppose that $X \sim Gamma(\alpha, \beta)$.
   a. If a is any positive or negative value such that $\alpha + a > 0$, show that $E(X^a) = \frac{\beta^a \Gamma(\alpha + a)}{\Gamma(\alpha)}$
   b. Why must $\alpha + a > 0$?
   c. Use the result from part (a) to give an expression for $E(\sqrt{X})$. What do you need to assume about α?
   d. Give an expression for $E\left(\frac{1}{\sqrt[3]{X^2}}\right)$. What do you need to assume about α here?