Lecture 19.5 – Agenda & Examples

Agenda

1. Review Questions
2. Correlation $\left(\rho = Corr(X_1, X_2) = \frac{Cov(X_1,X_2)}{\sqrt{Var[X_1]}\sqrt{Var[X_2]}}\right)$. What happens when we have independence?
3. Linear Combinations ($U_X = \sum_{i=1}^{n} a_i X_i$ and $U_Y = \sum_{j=1}^{m} a_j Y_j$
   a. Expectation → $E[U_X] = \sum_{i=1}^{n} a_i E[X_i]$
   b. Variance → $Var[U_X] = \sum_{i=1}^{n} a_i^2 Var[X_i] + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j Cov\left[X_i, X_j\right]$ for i≠j
   c. Covariance → $Cov[U_X, U_Y] = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j Cov\left[X_i, Y_j\right]$
4. Conditional Expectation
   a. Discrete → $E[g(X_1)|X_2 = x_2] = E_{1|2}[g(X_1)|X_2 = x_2] = \sum_{x_1 \epsilon S_1} g(x_1) p_{1|2}(x_1)$
   b. Continuous → → $E[g(X_1)|X_2 = x_2] = E_{1|2}[g(X_1)|X_2 = x_2] = \int_{-\infty}^{\infty} g(x_1) f_{1|2}(x_1) dx_1$
   c. $E[g(X_1)] = E\left[E[g(X_1)|X_2 = x_2]\right]$ with proof
   d. $V[g(X_1)] = E\left[V[g(X_1)|X_2 = x_2]\right] + V\left[E[g(X_1)|X_2 = x_2]\right]$ where $V\left[E[g(X_1)|X_2 = x_2]\right] = E[g(X_1)^2|X_2 = x_2] - (E[g(X_1)|X_2 = x_2])^2$
5. Examples

Review

1. Prove that $Cov[X_1 + X_2, Y] = Cov[X_1, Y] + Cov[X_2, Y]$
2. Consider the joint pmf $f(x_1, x_2) = (2^{x_1 - 1} 3^{x_2})^{-1}$ for $x_1, x_2 = 1,2, \ldots$
   a. Prove whether or not X_1 and X_2 are independent
   b. What is the $Cov(X_1, X_2)$?
3. Suppose a random variable X_1 has the following probability distribution

X_1 Values	-5	-2	0	2	5
Probabilities	.25	.2	.1	.2	.25

   Define $X_2 = X_1^2$. Show the following:
   a. $E[X_1] = E[X_1^3] = 0$
   b. $Cov(X_1, X_2) = 0$, implying that the random variables X_1 and X_2 are uncorrelated
   c. X_1 and X_2 are dependent random variables (showing that the property pertaining to independence and covariance only holds in one direction).

Lecture

1. Prove that if $Y_i = c_i + d_i X_i$, then $\rho_{Y_1,Y_2} = \rho_{X_1,X_2}$ for constants c_i, d_i
2. Let the discrete RV's Y_1 and Y_2 have the joint probability function $p(y_1, y_2) = \frac{1}{3}$ for $(y_1, y_2) = (-1,0), (0,1), (1,0)$.
   a. Find the $Cov[Y_1, Y_2]$ and ρ_{Y_1,Y_2}.
   b. Are Y_1 and Y_2 independent? (what do you notice about what correlation implies about independence? It only goes 1 way!)

3. (Text 5.113) A retail grocery merchant figures that her daily gain X from sales is a normally distributed RV with mean 50 and standard deviation 3 (measurement in dollars). X can be negative if she is forced to dispose of enough perishable goods. Also, she figures daily overhead costs Y to have a gamma distribution with alpha of 4 and beta of 2. If X and Y are independent, find the expected value and variance of her net daily gain. Would you expect her net gain for tomorrow to rise above $70?
4. Consider the following joint distribution
   $$f(x_1, x_2) = \frac{3}{2\pi\sqrt{8}} \exp\left(-\frac{9}{16}\left((x_1-2)^2 - \frac{2}{3}(x_1-2)(x_2-3) + (x_2-3)^2\right)\right) \quad x_1, x_2 \in \mathbb{R}$$

   Find $E[X_2^2 - 3X_2 | X_1 = 2]$.
5. Suppose that (U_1, U_2) is distributed as $N_2(5, 15, 8, 8, \rho)$ for some $\rho \in (-1,1)$. Let $X_1 = U_1 + U_2$ and $X_2 = U_1 - U_2$. Show that X_1 and X_2 are uncorrelated.
6. Textbook Problem 5.142